Numerical Calculation of the Potential Distribution in Ion Slit Lens Systems III

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In previous papers [1, 2] the potential distribution was calculated in ion slit lens systems, consisting of three slits in three parallel electrodes and satisfying certain conditions concerning their shape.

In the present paper the computing methods are generalized to slit systems of an arbitrary number of electrodes, with as the only restriction, that slits broader than the distances to neighbouring slits are separated by slits, narrower than the respective distance, and that a pair of electrodes with a mutual distance smaller than their slit widths are separated from the neighbouring slits by distances greater than the respective slit widths.

For slit systems, satisfying this condition the parameters are computed, necessary to perform the Schwarz-Christoffel transformation. Formulae are given to compute the potential distribution and field strength. In a typical example the potential distribution and field strength are computed in the region around two parallel electrodes with broad slits compared with the distance between the electrodes.

In a previous paper [1] methods were developed to calculate the parameters necessary to compute the potential distribution in certain three slit lens systems. The conditions to the slit system were:

1. parallel electrodes,
2. infinite slit lenghts,
3. electrodes thicknesses very small, compared with the electrode distances and slit widths,
4. all centres of the slits in one plane perpendicular to the electrodes.

Under the same assumptions the calculations can be performed for a system with a number of N electrodes.

In Fig. 1 the cross section is drawn of such a system with a plane perpendicular to the direction of the slits.

The slit widths are denoted successively by $2\,s_1$, $2\,s_2$, ..., $2\,s_N$, the electrode distances by $\pi\,r_1$, $\pi\,r_2$, ..., $\pi\,r_{N-1}$.

For the calculations the following procedure is adopted. The quantities a_1, a_2, a_3 etc. are attributed successively to the internal and infinite ends of the electrodes, as indicated in Fig. 1. These a_m are the parameters appearing in the formulae (5, 6) for the calculation of potential and field strength around the electrodes. They are found as the solution of the system of $2\,N-1$ equations [1]:

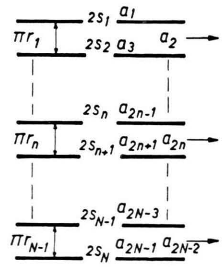

Fig. 1. Cross section of the slit system.

$$2\,s_n=\int\limits_{-a_{2n-1}}^{+a_{2n-1}}\frac{(s^2-a_1^2)\,(s^2-a_3^2)\dots(s^2-a_{2n-1}^2)\,(s^2-a_{2n+1}^2)\dots(s^2-a_{2N-1}^2)}{(s^2-a_2^2)\,(s^2-a_4^2)\dots(s^2-a_{2n}^2)\,(s^2-a_{2n+2}^2)\dots s^2}\,\mathrm{d}s \tag{1}$$

$$=2\,a_{2n-1}+r_1\ln\frac{a_2+a_{2n-1}}{a_2-a_{2n-1}}+r_2\ln\frac{a_4+a_{2n-1}}{a_4-a_{2n-1}}+\dots+r_{n-1}\ln\frac{a_{2n-2}+a_{2n-1}}{a_{2n-2}-a_{2n-1}}$$

$$+r_n\ln\frac{a_{2n-1}+a_{2n}}{a_{2n-1}-a_{2n}}+\dots+r_{N-1}\ln\frac{a_{2n-1}+a_{2N-2}}{a_{2n-1}-a_{2N-2}}+2\,\frac{a_1^2\,a_3^2\dots a_{2n-3}^2\,a_{2n-1}\,a_{2n+1}^2\dots a_{2N-1}^2}{a_2^2\,a_4^2\dots a_{2n-2}^2\,a_{2n}^2\dots a_{2N-2}^2},$$

$$\pi\,r_n=\Im\mathrm{m}\int\limits_{a_{2n+1}}^{a_{2n-1}}\frac{(s^2-a_1^2)\,(s^2-a_3^2)\dots(s^2-a_{2n-1}^2)\,(s^2-a_{2n+1}^2)\dots(s^2-a_{2N-1}^2)}{(s^2-a_2^2)\,(s^2-a_4^2)\dots(s^2-a_{2n}^2)\,(s^2-a_{2n+2}^2)\dots s^2}\,\mathrm{d}s \tag{2}$$

$$=\pi\,\frac{(a_1^2-a_{2n}^2)\,(a_3^2-a_{2n}^2)\dots(a_{2n-1}^2-a_{2n}^2)\,(a_{2n}^2-a_{2n+1}^2)\dots(a_{2n}^2-a_{2N-1}^2)}{(a_2^2-a_{2n}^2)\,(a_4^2-a_{2n}^2)\dots(2a_{2n})\,(a_{2n}^2-a_{2n+2}^2)\dots a_{2n}^2}.$$

[1] A. J. H. Boerboom, Z. Naturforschg. **14 a**, 809 [1959].

[2] A. J. H. Boerboom, Z. Naturforschg. **15 a**, 244 [1960].

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The only solution of this system, with a physical meaning, is the one in which all a_m are real and

$$a_m > a_{m+1} > 0\,. \tag{3}$$

From the quantities a_m the potential distribution and field strength are found in the following way:

The complex parameter $w = u + i\,v$ is introduced, connected to the coordinates x and y in the slit system through the formula

$$z = x + i\,y = w + r_1 \ln \frac{w+a_2}{w-a_2} + r_2 \ln \frac{w+a_4}{w-a_4} + \dots + r_n \ln \frac{w+a_{2n}}{w-a_{2n}} + \dots + r_{N-1} \ln \frac{w+a_{2N-2}}{w-a_{2N-2}} + \frac{a_1^2\, a_3^2 \dots a_{2N-1}^2}{a_2^2\, a_4^2 \dots a_{2N-2}^2}\, \frac{1}{w}\,. \tag{4}$$

Here the main values of the logarithms are to be taken.

The potential is computed as a linear superposition of the contributions of all electrodes. The contribution of the n^{th} half-electrode is given by

$$V_n(u,v) = \frac{V_n}{\pi} \left\{ \arctan \frac{v}{u - a_{2n-2}} - \arctan \frac{v}{u - a_{2n}} \right\} \tag{5}$$

with V_n = potential of the n^{th} half-electrode.

The field strength is found by differentiation. Along the main axis the contribution of the n^{th} half-electrode is given by the formulae

$$\frac{\partial V_n(u,v)}{\partial y} = \frac{\partial V_n(u,v)}{\partial v} \Big/ \frac{\partial y}{\partial v}\,, \quad \frac{\partial V_n(u,v)}{\partial x} = \frac{\partial V_n(u,v)}{\partial u} \Big/ \frac{\partial u}{\partial x}\,, \tag{6}$$

$$\frac{\partial y}{\partial v} = 1 + 2\, r_1 \frac{a_2}{v^2 + a_2^2} + \dots + 2\, r_n \frac{a_{2n}}{v^2 + a_{2n}^2} + \dots + 2\, r_{N-1} \frac{a_{2N-2}}{v^2 + a_{2N-2}^2} + \frac{a_1^2\, a_3^2 \dots a_{2N-1}^2}{a_2^2\, a_4^2 \dots a_{2N-2}^2}\, \frac{1}{v^2} = \frac{\partial x}{\partial u}\,,$$

$$\frac{\partial V_n(u,v)}{\partial v} = \frac{V_n}{\pi} \left\{ \frac{a_{2n}}{v^2 + a_{2n}^2} - \frac{a_{2n-2}}{v^2 + a_{2n-2}^2} \right\}, \quad \frac{\partial V_n(u,v)}{\partial u} = \frac{V_n}{\pi} \left\{ \frac{v}{v^2 + a_{2n}^2} - \frac{v}{v^2 + a_{2n-2}^2} \right\}.$$

In addition to the restriction (3) we find that (see section 1)

$$\begin{aligned} a_{2n-1}^2 &\gg a_{2n}^2 \quad \text{if} \quad 2\, s_n < \pi r_n\,, \\ a_{2n}^2 &\gg a_{2n+1}^2 \quad \text{if} \quad 2\, s_{n+1} < \pi\, r_n\,. \end{aligned} \tag{7}$$

In general, the equations (1, 2) can be solved by the following methods if a set of two or three subsequent $a_m{}^2$ of the same order of magnitude is separated from similar sets by at least one order of magnitude. This means, that the potential and field strength can be computed by the following methods, if slits, broader than the distances to the neighbouring electrodes are separated from the rest of the slit system by slits narrower than these distances (Fig. 2a) and pairs of electrodes with a mutual distance smaller than their slit widths are separated from the other slits by distances greater than the slit widths (Fig. 2 b).

In the previous paper[1] the case $N = 3$ and $a_m^2 \ll a_{m+1}^2$ was treated; in the paper[2] $N = 3$, no restrictions to the $a_m{}^2$ were made but is was supposed, that $s_1 = s_3$ and $r_1 = r_2$. In the next paragraphs several cases are treated concerning the orders of magnitude of the $a_m{}^2$, with N arbitrary.

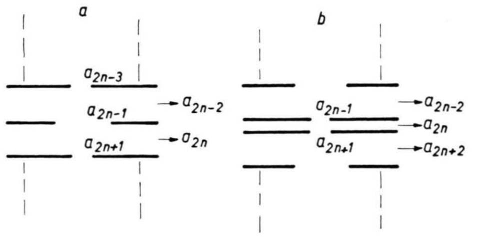

Fig. 2. The orders of magnitude of the $a_m{}^2$.

1. Solution of the equations (1,2) for the case

$$a_m^2 \gg a_{m+1}^2$$

With the assumption

$$a_m^2 \gg a_{m+1}^2 \quad \text{for} \quad m = 1, 2, \dots, 2\,N-2\,, \tag{8}$$

and neglecting terms of higher order of magnitude, the equations (1, 2) can be reduced. The integral appearing in equation (1) can be evaluated in the complex s-plane along a circle around the origin:

$$s = a_{2n-1} \exp\{i\,\varphi\}\,, \qquad -\pi \le \varphi \le 0\,.$$

Along the whole integration path

$$\begin{aligned} |s^2 - a_m^2| &\gg s^2 \quad \text{if} \quad m < 2n-1, \\ |s^2 - a_m^2| &\gg a_m^2 \quad \text{if} \quad m > 2n-1, \end{aligned} \tag{9}$$

so in first approximation

$$2 s_n \approx \int_{-a_{2n-1}}^{+a_{2n-1}} \frac{(-a_1^2)(-a_3^2)\dots(-a_{2n-3}^2)(s^2-a_{2n-1}^2)\, s^2 s^2 \dots}{(-a_2^2)(-a_4^2)\dots(-a_{2n-2}^2)\, s^2 s^2 s^2 \dots}\, ds$$

$$= \frac{a_1^2 a_3^2 \dots a_{2n-3}^2}{a_2^2 a_4^2 \dots a_{2n-2}^2} \int_{-a_{2n-1}}^{+a_{2n-1}} \frac{s^2 - a_{2n-1}^2}{s^2}\, ds = \frac{a_1^2 a_3^2 \dots a_{2n-3}^2}{a_2^2 a_4^2 \dots a_{2n-2}^2} \cdot 4\, a_{2n-1}. \tag{10}$$

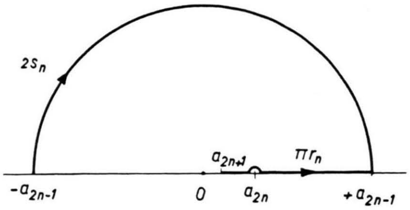

Fig. 3. Integration paths in the complex s-plane.

The integral in equation (2) is evaluated along the real axis of the s-plane, except for a small semi-circle around the singular point $s = a_{2n}$:

$$s - a_{2n} = r \exp\{i\,\varphi\}, \quad -\pi \leq \varphi \leq 0.$$

The parts of the integral along the real axis give no imaginary contribution, so only the part of the integral along the semi-circle remains. The radius r of this semi-circle can be taken very small, so along this path

$$\begin{aligned} |s^2 - a_m^2| &\gg s^2 \quad \text{if} \quad m < 2n, \\ |s^2 - a_m^2| &\gg a_m^2 \quad \text{if} \quad m > 2n, \end{aligned} \tag{11}$$

and in first approximation:

$$\pi r_n \approx \Im\mathfrak{m} \int_{a_{2n+1}}^{a_{2n-1}} \frac{(-a_1^2)(-a_3^2)\dots(-a_{2n-1}^2)\, s^2 \dots s^2}{(-a_2^2)(-a_4^2)\dots(s^2 - a_{2n}^2)\, s^2 \dots s^2}\, ds\,.$$

Now $\Im\mathfrak{m} \int_{a_{2n}-r}^{a_{2n}+r} \frac{ds}{s^2 - a_{2n}^2} = \Im\mathfrak{m} \int_{a_{2n}-r}^{a_{2n}+r} \frac{1}{2\, a_{2n}} \left(\frac{1}{s + a_{2n}} - \frac{1}{s - a_{2n}} \right) ds,$

and the first term gives no imaginary contribution to the integral. The second term gives

$$\frac{-1}{2\, a_{2n}}\, \Im\mathfrak{m} \int_{-\pi}^{0} \frac{r\, d \exp\{i\,\varphi\}}{r \exp\{i\,\varphi\}} = -\frac{\pi}{2\, a_{2n}},$$

$$\text{so} \quad r_n \approx \frac{a_1^2 a_3^2 \dots a_{2n-3}^2}{a_2^2 a_4^2 \dots a_{2n-2}^2}\, \frac{a_{2n-1}^2}{2 a_{2n}}. \tag{12}$$

The exact equations (1, 2) are reduced to the approximate equations (10, 12) which have the solution given by the recursion formulae

$$a_1 = \frac{s_1}{2}, \quad \frac{a_{2n-1}}{a_{2n-2}} = \frac{s_n}{4\, r_{n-1}} = v_{2n-2}, \qquad \frac{a_{2n}}{a_{2n-1}} = \frac{s_n}{4\, r_n} = v_{2n-1} \tag{13}$$

$$\text{or} \quad a_n = \frac{s_1}{2} v_1 v_2 \dots v_{n-1}. \tag{14}$$

The approximate equations were deduced under the assumption (8). This corresponds to the condition $v_m{}^2 \ll 1$, which means that

$$\left(\frac{2\, s_n}{\pi\, r_{n-1}} \right)^2 \ll \left(\frac{8}{\pi} \right)^2 \text{ and } \left(\frac{2\, s_n}{\pi\, r_n} \right)^2 \ll \left(\frac{8}{\pi} \right)^2. \tag{15}$$

We may attach the following meaning to the sign $\ll$: if two quantities p and q are connected through $p \ll q$, $p/q < 0.1$. Then (15) implies that all slit widths of the system have to be smaller than 0.8 times the distances to the neighbouring electrodes.

As N denotes the total number of electrodes, r_N can be considered to be infinite and the formulae (13) give $a_{2N} = 0$. If the N^{th} electrode has no slit, $2\, s_N$ is zero, and $a_{2N-1} = 0$. In the latter case one sees from formulae (4, 5), that only the a_m with m = even have to be calculated.

2. Higher approximations in the case $a_m^2 \gg a_{m+1}^2$

Higher approximations are found by more accurate integration of the integrals in (1, 2). In (1) the factors $(s^2 - a_{2n-2}^2)$ and $(s^2 - a_{2n}^2)$ are taken into consideration; in (2) the factors $(s^2 - a_{2n-1}^2)$ and $(s^2 - a_{2n+1}^2)$. The integration gives in second approximation:

$$s_n = 2\, a_{2n-1} \frac{a_1^2 a_3^2 \dots a_{2n-3}^2}{a_2^2 a_4^2 \dots a_{2n-2}^2} \left\{ 1 - \tfrac{1}{3} v_{2n-2}^2 - \tfrac{1}{3} v_{2n-1}^2 \right\},$$

$$r_n = \frac{a_{2n-1}^2}{2 a_{2n}} \frac{a_1^2 a_3^2 \dots a_{2n-3}^2}{a_2^2 a_4^2 \dots a_{2n-2}^2} \left\{ 1 - v_{2n-1}^2 - v_{2n}^2 \right\}. \tag{16}$$

In the first of these equations, for $n = 1$ v_0 has to be taken zero.

$$\text{Putting} \quad a_n = \frac{s_1}{2} v_1 v_2 \dots v_{n-1} \left\{ 1 + \sum A_{nm} v_m^2 \right\}, \tag{17}$$

we get the following scheme for the A_{nm} (see p. 256). We supposed $v_m{}^2 \ll 1$, but if $n \gg 1$, $1 + \Sigma\, A_{nm} v_m{}^2$ could get negative. As we require $a_n > 0$, it means, that it is no longer allowed to neglect the higher order terms. Therefore, for large values of n it is better to use the formulae (16), or to use the recur-

	v_1^2	v_2^2	v_3^2	v_4^2	...	v_{2n-3}^2	v_{2n-2}^2	v_{2n-1}^2	v_{2n}^2	...	v_{2N-2}^2
A_1	+1/3	0	0	0	...	0	0	0	0	...	0
A_2	−1/3	−1	0	0	...	0	0	0	0	...	0
A_3	−4/3	−5/3	+1/3	0	...	0	0	0	0	...	0
A_4	−4/3	−4/3	−1/3	−1	...	0	0	0	0	...	0
.											
.											
.											
A_{2n-1}	−4/3	−4/3	−4/3	−4/3	...	−4/3	−5/3	+1/3	0	...	0
A_{2n}	−4/3	−4/3	−4/3	−4/3	...	−4/3	−4/3	−1/3	−1	...	0
A_{2N-2}	−4/3	−4/3	−4/3	−4/3	...	−4/3	−4/3	−4/3	−4/3	...	−1

sion formulae derived therefrom:

$$\left.\begin{aligned} a_1 &= \frac{s_1}{2}\left(1+\tfrac{1}{3}v_1^2\right), \\ \frac{a_{2n-1}}{a_{2n-2}} &= v_{2n-2}\left\{1 - v_{2n-3}^2 - \tfrac{2}{3}v_{2n-2}^2 + \tfrac{1}{3}v_{2n-1}^2\right\}, \\ \frac{a_{2n}}{a_{2n-1}} &= v_{2n-1}\left\{1 + \tfrac{1}{3}v_{2n-2}^2 - \tfrac{2}{3}v_{2n-1}^2 - v_{2n}^2\right\}. \end{aligned}\right\} \quad (18)$$

However, introducing "adapted" slit widths and electrode distances defined through

$$\left.\begin{aligned} S_1 &= s_1\left(1+\tfrac{1}{3}v_1^2\right), \\ S_n &= s_n\left(1+\tfrac{1}{3}v_{2n-2}^2+\tfrac{1}{3}v_{2n-1}^2\right), \\ R_n &= r_n\left(1+v_{2n-1}^2+v_{2n}^2\right), \end{aligned}\right\} \quad (19)$$

the solution (13) becomes in second approximation

$$a_1 = \frac{S_1}{2}, \quad \frac{a_{2n-1}}{a_{2n-2}} = \frac{S_n}{4\,R_{n-1}}, \quad \frac{a_{2n}}{a_{2n-1}} = \frac{S_n}{4\,R_n}. \quad (20)$$

Due to the particular shape of the equations (1, 2) the definitions (19) can be extended with higher order terms, to get the exact values of the a_m, given by (20).

3. Solution of the equations (1,2) when some a_m^2 are of the same order of magnitude

When some subsequent quantities a_m, $a_{m+1}\ldots$ have squares with the same order of magnitude, this has to be taken into consideration. The relative factors $(s^2 - a_m^2)\ldots$ cannot be reduced to either s^2 or $(-a_m^2)$, on integrating the equations (1, 2). We distinguish several cases.

First case: $\ldots a_{2n-2}^2 \gg a_{2n-1}^2 \gtrapprox a_{2n}^2 \gg a_{2n+1}^2 \ldots$.

Integration of the equations (1, 2) along the paths defined in 1., gives in first approximation:

$$s_n = \frac{a_1^2\ldots a_{2n-3}^2}{a_2^2\ldots a_{2n-2}^2}\left\{a_{2n-1} + \frac{a_{2n-1}^2 - a_{2n}^2}{2a_{2n}}\ln\frac{a_{2n-1}+a_{2n}}{a_{2n-1}-a_{2n}}\right\},$$

$$r_n = \frac{a_1^2\ldots a_{2n-3}^2}{a_2^2\ldots a_{2n-2}^2}\;\frac{a_{2n-1}^2 - a_{2n}^2}{2a_{2n}}.$$

We introduce $a_{2n-1}/a_{2n} = w_{2n-1}$ and eliminate all a_m:

$$\frac{s_n}{r_n} = \frac{2\,w_{2n-1}}{w_{2n-1}^2 - 1} + \ln\frac{w_{2n-1}+1}{w_{2n-1}-1} = F(w_{2n-1}). \quad (21)$$

The function in the right hand side of this equation is plotted and tabulated in Fig. 4 resp. Table 1, so

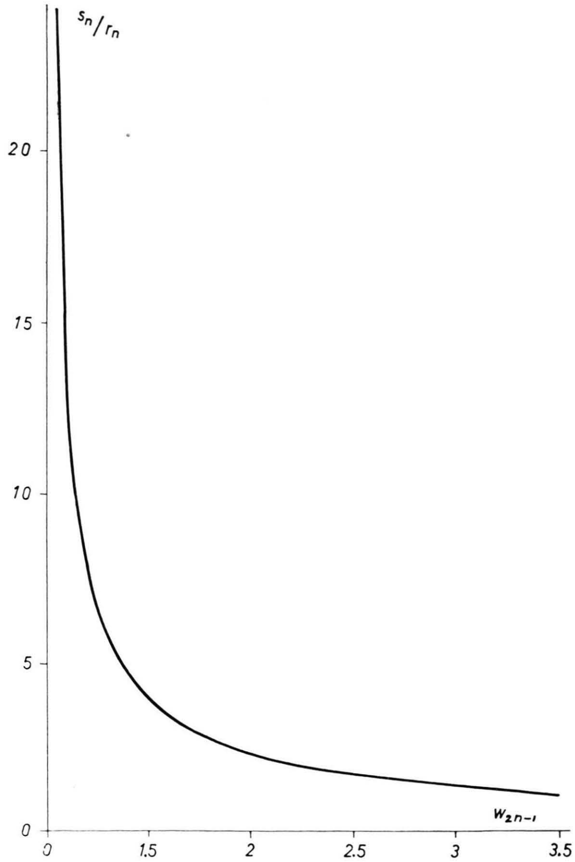

Fig. 4. The function F of equation (21).

w_{2n-1} can be solved from this equation. If the $a_1\ldots a_{2n-2}$ are known from the other equations, a_{2n-1} and a_{2n} can be found.

w_{2n-1}	$F(w_{2n-1})$	w_{2n-1}	$F(w_{2n-1})$
1.01	105.80	1.30	5.805
1.02	55.11	1.50	4.009
1.03	38.04	2.00	2.432
1.05	24.20	2.50	1.7997
1.10	13.521	3.00	1.4432
1.20	7.853	3.50	1.2100

Table 1.

Second case: $\ldots a_{2n-1}^2 \gg a_{2n}^2 \gtrapprox a_{2n+1}^2 \gg a_{2n+2}^2 \cdots$.

Integration of (1, 2) gives in first approximation:

$$s_{n+1} = \frac{a_1^2 \ldots a_{2n-1}^2}{a_2^2 \ldots a_{2n}^2} \left\{ a_{2n+1} + \frac{a_{2n}^2 - a_{2n+1}^2}{2a_{2n}} \ln \frac{a_{2n} + a_{2n+1}}{a_{2n} - a_{2n+1}} \right\},$$

$$r_n = \frac{a_1^2 \ldots a_{2n-1}^2}{a_2^2 \ldots a_{2n}^2} \, \frac{a_{2n}^2 - a_{2n+1}^2}{2a_{2n}} .$$

Substituting $a_{2n}/a_{2n+1} = w_{2n}$ and eliminating $a_1 \ldots a_{2n+1}$ we get

$$\frac{s_{n+1}}{r_n} = \frac{2w_{2n}}{w_{2n}^2 - 1} + \ln \frac{w_{2n} + 1}{w_{2n} - 1} . \qquad (22)$$

w_{2n} can be found from the same plot or table as before, and again a_{2n} and a_{2n+1} can be found if all preceeding a_m are known.

Third case:

$$\ldots a_{2n-2}^2 \gg a_{2n-1}^2 \gtrapprox a_{2n}^2 \gtrapprox a_{2n+1}^2 \gg a_{2n+2}^2 \cdots .$$

Taking into consideration all significant factors the integration of (1, 2) provides in first approximation:

$$s_n = \frac{a_1^2 \ldots a_{2n-3}^2}{a_2^2 \ldots a_{2n-2}^2} \left\{ a_{2n-1} + \frac{(a_{2n-1}^2 - a_{2n}^2)(a_{2n}^2 - a_{2n+1}^2)}{2a_{2n}^3} \ln \frac{a_{2n-1} + a_{2n}}{a_{2n-1} - a_{2n}} + \frac{a_{2n-1} a_{2n+1}^2}{a_{2n}^2} \right\},$$

$$r_n = \frac{a_1^2 \ldots a_{2n-3}^2}{a_2^2 \ldots a_{2n-2}^2} \, \frac{(a_{2n-1}^2 - a_{2n}^2)(a_{2n}^2 - a_{2n+1}^2)}{2a_{2n}^3},$$

$$s_{n+1} = \frac{a_1^2 \ldots a_{2n-3}^2}{a_2^2 \ldots a_{2n-2}^2} \left\{ a_{2n+1} + \frac{(a_{2n-1}^2 - a_{2n}^2)(a_{2n}^2 - a_{2n+1}^2)}{2a_{2n}^3} \ln \frac{a_{2n} + a_{2n+1}}{a_{2n} - a_{2n+1}} + \frac{a_{2n-1}^2}{a_{2n}^2} a_{2n+1} \right\}.$$

Substituting $a_{2n-1}/a_{2n} = w_{2n-1}$ and $a_{2n}/a_{2n+1} = w_{2n}$ all a_m can be eliminated and one gets two equations in the two unknowns w_{2n-1} and w_{2n}:

$$\frac{s_n}{r_n} = \frac{2\,w_{2n-1}}{w_{2n-1}^2 - 1} \, \frac{w_{2n}^2 + 1}{w_{2n}^2 - 1} + \ln \frac{w_{2n-1} + 1}{w_{2n-1} - 1} = F_1(w_{2n-1}, w_{2n}),$$

$$\frac{s_{n+1}}{r_n} = \frac{2\,w_{2n}}{w_{2n}^2 - 1} \, \frac{w_{2n-1}^2 + 1}{w_{2n-1}^2 - 1} + \ln \frac{w_{2n} + 1}{w_{2n} - 1} = F_1(w_{2n}, w_{2n-1}). \qquad (23)$$

The unknowns w_{2n-1} and w_{2n} can be found from the plot in Fig. 5 or from Table 2. If $a_1 \ldots a_{2n-2}$ are known, a_{2n-1}, a_{2n} and a_{2n+1} can be computed.

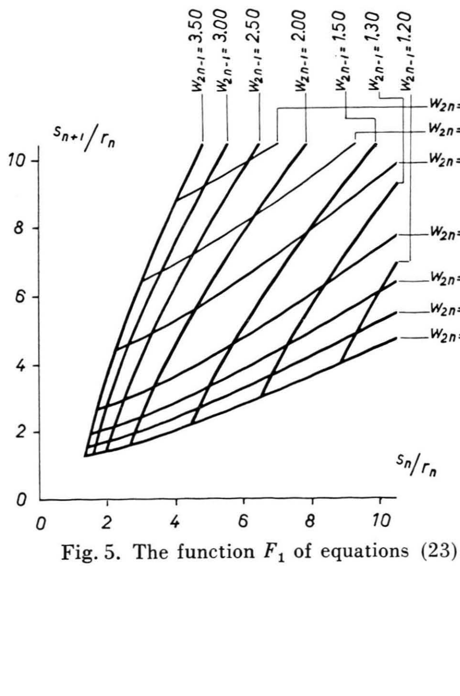

Fig. 5. The function F_1 of equations (23).

Fourth case:

$$\ldots a_{2n-3}^2 \gg a_{2n-2}^2 \gtrapprox a_{2n-1}^2 \gtrapprox a_{2n}^2 \gg a_{2n+1}^2 \cdots .$$

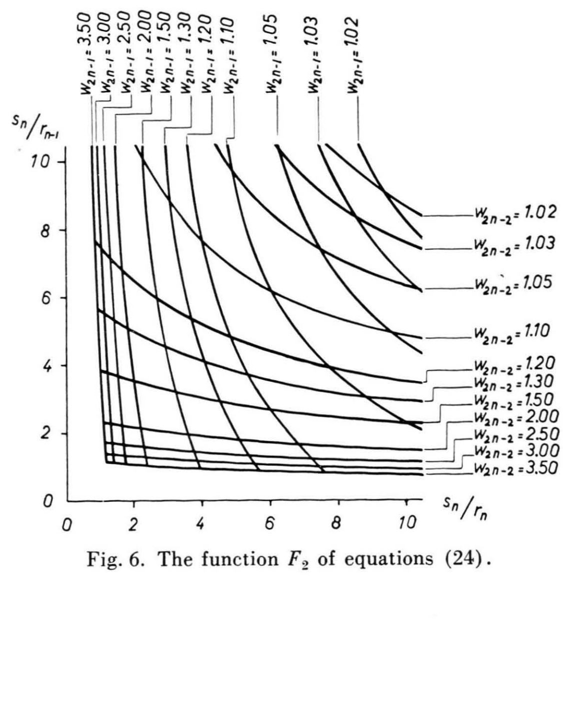

Fig. 6. The function F_2 of equations (24).

Table 2. The function $\frac{s_n}{r_n} = \frac{2w_{2n-1}}{w_{2n-1}^2 - 1} \frac{w_{2n}^2 + 1}{w_{2n}^2 - 1} + \ln \frac{w_{2n-1} + 1}{w_{2n-1} - 1} = F_1(w_{2n-1}, w_{2n})$.

w_{2n} \ w_{2n-1}	1.01	1.02	1.03	1.05	1.10	1.20	1.30	1.50	2.00	2.50	3.00	3.50
1.01	10106	5079	3404	2063	1055.9	550.6	380.7	242.8	135.10	96.56	76.07	63.12
1.02	5081	2555	1712.6	1038.4	532.1	277.9	192.35	122.82	68.44	48.95	38.57	32.01
1.03	3406	1713.4	1148.9	697.0	357.6	186.98	129.55	82.83	46.22	33.08	26.07	21.64
1.05	2067	1040.4	698.1	424.0	217.9	114.28	79.33	50.84	28.45	20.38	16.077	13.351
1.10	1062.9	536.0	360.2	219.3	113.29	59.80	41.69	26.87	15.130	10.870	8.586	7.136
1.20	562.6	284.6	191.79	117.33	61.14	32.65	22.93	14.919	8.493	6.129	4.852	4.038
1.30	397.1	201.5	136.09	83.59	43.89	23.66	16.727	10.966	6.297	4.560	3.617	3.014
1.50	266.6	135.90	92.16	56.98	30.28	16.580	11.834	7.849	4.565	3.323	2.643	2.206
2.00	172.80	88.77	60.59	37.86	20.50	11.489	8.317	5.609	3.321	2.435	1.943	1.6248
2.50	144.09	74.35	50.93	32.01	17.512	9.930	7.240	4.924	2.940	2.162	1.7289	1.4471
3.00	130.93	67.73	46.50	29.32	16.140	9.216	6.747	4.609	2.765	2.038	1.6307	1.3656
3.50	123.67	64.09	44.05	27.84	15.383	8.822	6.475	4.436	2.669	1.9690	1.5765	1.3206

Table 3. The function $\frac{s_n}{r_n} = \frac{w_{2n-1}}{w_{2n-2}} \frac{w_{2n-2}^2 - 1}{w_{2n-1}^2 - 1} \ln \frac{w_{2n-2} + 1}{w_{2n-2} - 1} + \ln \frac{w_{2n-1} + 1}{w_{2n-1} - 1} = F_2(w_{2n-1}, w_{2n-2})$.

w_{2n-1} \ w_{2n-2}	1.01	1.02	1.03	1.05	1.10	1.20	1.30	1.50	2.00	2.50	3.00	3.50
1.01	10.607	14.489	17.825	23.52	34.51	49.48	59.63	72.70	88.11	94.71	98.18	100.24
1.02	7.280	9.230	10.907	13.768	19.290	26.81	31.91	38.48	46.22	49.54	51.28	52.32
1.03	6.000	7.306	8.429	10.346	14.045	19.085	22.50	26.90	32.09	34.31	35.48	36.17
1.05	4.795	5.586	6.266	7.427	9.668	12.720	14.788	17.453	20.59	21.94	22.65	23.07
1.10	3.597	4.002	4.350	4.943	6.089	7.650	8.707	10.070	11.676	12.365	12.727	12.941
1.20	2.686	2.896	3.078	3.387	3.983	4.796	5.346	6.056	6.892	7.251	7.439	7.551
1.30	2.236	2.381	2.506	2.720	3.132	3.693	4.074	4.564	5.142	5.389	5.519	5.596
1.50	1.7361	1.8288	1.9085	2.044	2.307	2.664	2.907	3.219	3.587	3.745	3.828	3.877
2.00	1.1690	1.2205	1.2647	1.3403	1.4861	1.6848	1.8193	1.9928	2.197	2.285	2.331	2.358
2.50	0.8976	0.9343	0.9660	1.0199	1.1241	1.2660	1.3621	1.4860	1.6320	1.6946	1.7275	1.7470
3.00	0.7327	0.7617	0.7866	0.8291	0.9111	1.0229	1.0986	1.1961	1.3111	1.3604	1.3863	1.4016
3.50	0.6206	0.6447	0.6653	0.7006	0.7686	0.8613	0.9241	1.0051	1.1005	1.1414	1.1628	1.1756

Table 4. Potential and field strength along the main axis of the slit system of Fig. 6.

y	$V(y)$ (1)	$V(y)$ (2)	$\frac{\partial V}{\partial y}$ (1)	$\frac{\partial V}{\partial y}$ (2)	$\frac{\partial V}{\partial x}$ (3)
− 21.88	0.01000	0.01360	0.00046	0.00062	0.000007
− 10.91	0.02000	0.02718	0.00182	0.00247	0.000053
− 4.263	0.05000	0.06776	0.01112	0.01495	0.00080
− 1.956	0.10000	0.13401	0.04131	0.05374	0.00574
− 0.638	0.2000	0.2571	0.13112	0.15082	0.03222
+ 0.583	0.5000	0.5210	0.3982	0.25294	0.13511
1.502	1.0000	0.7213	0.6833	0.16701	0.17842
2.744	2.000	0.85366	0.88859	0.06275	0.13408
5.897	5.000	0.94059	0.97984	0.01158	0.06184
10.95	10.000	0.97023	0.99487	0.00296	0.03160
20.97	20.00	0.98511	0.99871	0.00074	0.01589

In the same way we get:

$$r_{n-1} = \frac{a_1^2 \ldots a_{2n-5}^2}{a_2^2 \ldots a_{2n-4}^2} \frac{a_{2n-3}^2}{2a_{2n-2}} \frac{a_{2n-2}^2 - a_{2n-1}^2}{a_{2n-2}^2 - a_{2n}^2},$$

$$s_n = \frac{a_1^2 \ldots a_{2n-5}^2}{a_2^2 \ldots a_{2n-4}^2} \left\{ \frac{a_{2n-3}^2}{2a_{2n-2}} \frac{a_{2n-2}^2 - a_{2n-1}^2}{a_{2n-2}^2 - a_{2n}^2} \ln \frac{a_{2n-2} + a_{2n-1}}{a_{2n-2} - a_{2n-1}} + \frac{a_{2n-3}^2}{2a_{2n}} \frac{a_{2n-1}^2 - a_{2n}^2}{a_{2n-2}^2 - a_{2n}^2} \ln \frac{a_{2n-1} + a_{2n}}{a_{2n-1} - a_{2n}} \right\},$$

$$r_n = \frac{a_1^2 \ldots a_{2n-5}^2}{a_2^2 \ldots a_{2n-4}^2} \frac{a_{2n-3}^2}{2a_{2n}} \frac{a_{2n-1}^2 - a_{2n}^2}{a_{2n-2}^2 - a_{2n}^2},$$

$$\frac{s_n}{r_{n-1}} = \ln\frac{w_{2n-2}+1}{w_{2n-2}-1} + \frac{w_{2n-2}}{w_{2n-1}}\,\frac{w_{2n-1}^2-1}{w_{2n-2}^2-1}\ln\frac{w_{2n-1}+1}{w_{2n-1}-1} = F_2(w_{2n-2}, w_{2n-1}),$$
$$\frac{s_n}{r_n} = \frac{w_{2n-1}}{w_{2n-2}}\,\frac{w_{2n-2}^2-1}{w_{2n-1}^2-1}\ln\frac{w_{2n-2}+1}{w_{2n-2}-1} + \ln\frac{w_{2n-1}+1}{w_{2n-1}-1} = F_2(w_{2n-1}, w_{2n-2}). \qquad (24)$$

These last functions are plotted and tabulated in Fig. 6 and Table 3.

If assumptions are made concerning the order of magnitude of the adjacent a_m^2 for these four cases, higher order approximations can be found in a similar way as shown before [2].

5. Numerical example

As an example of the above theory the potential and field strength are computed in the electrode configuration of Fig. 7, in which $2\,s_1 = 2\,s_2 = 3$ and $\pi\,r_1 = 1$. In this case $a_1^2 = a^2 \approx a_2^2 = b^2 \approx a_3^2 = c^2$, so we use the formulae (23) of 4, third case.

$\frac{s_1}{r_1} = \frac{s_2}{r_1} = 4.7124$ and $w_1 = w_2 = 1.741$.

We find $a = 0.8144$, $b = 0.4678$, $c = 0.2687$.

More accurate figures are

$a = 0.814718$, $b = 0.468027$, $c = 0.268865$.

With the latter values for a, b, and c the potential distribution and field strength along the main axis were computed, using the formulae (4, 5, 6) for the following cases:

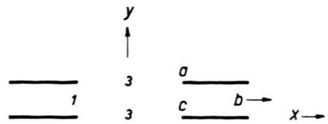

Fig. 7. Numerical example.

(1) Both electrodes on zero potential, but an external field of unit field strength right handside of the lens.

(2) First electrode on potential $+1$, second electrode zero.

(3) First half-electrode on potential $+1$, the other half of the first electrode and second electrode zero.

In the latter case only the cross field was computed. The results are given in Table 4 and plotted in Fig. 8.

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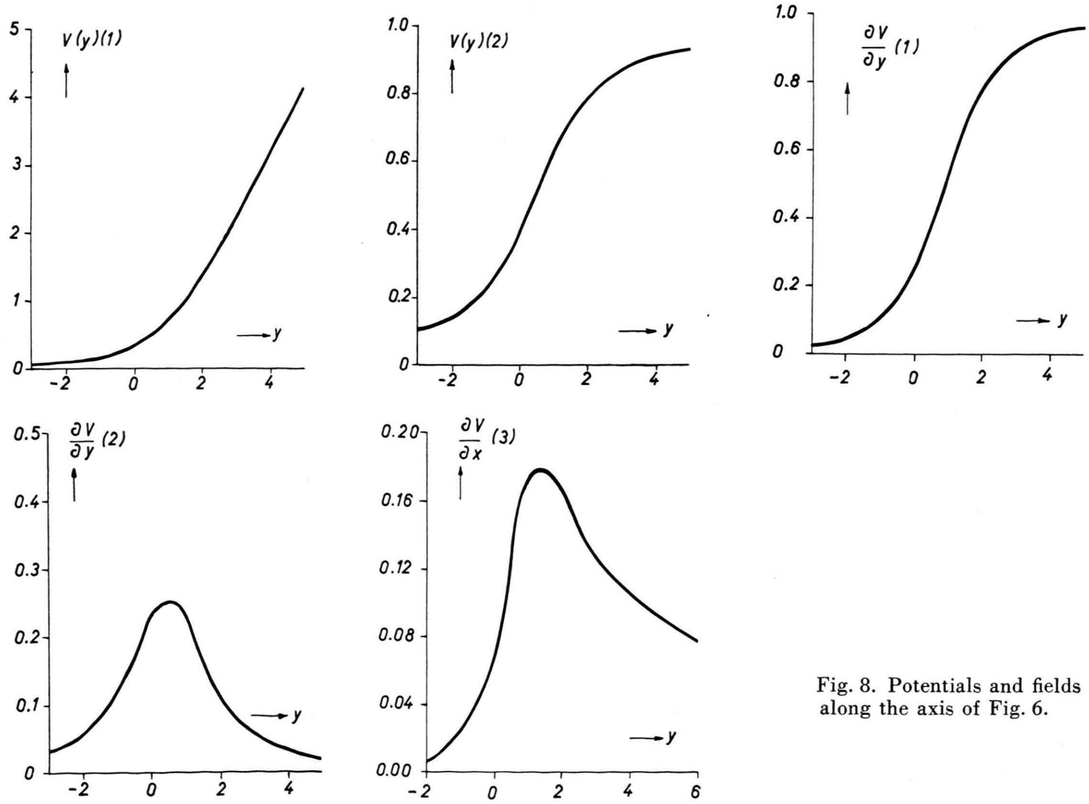

Fig. 8. Potentials and fields along the axis of Fig. 6.